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THE THEORY OF THE CONTROL OF A MONOCYCLE†

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The forward motion of a monocycle is studied in terms of a mechanical model: a cylinder with an inverted pendulum attached to it by a hinge. The rotation of the pendulum about the cylinder is implemented by an electric drive. The monocycle can be moved (rolled) over a surface with the help of the same drive. The control parameter is a voltage of bounded magnitude. A control law (with saturation), linear in the phase coordinates, is constructed, under which the pendulum is stabilized in the upper unstable equilibrium position, while the monocycle is maintained in position or moved. The domain of attraction of the desired steady state may be made the maximum possible (in the linear approximation). The domain of stability of the equilibrium state of the system is constructed in the plane of the following parameters: the total feedback gain and the delay. Delay appears in the control loop when there is an inductance in the rotor circuit of the motor.

The monocycle is of interest as an object of both theoretical and applied research. A new means of transportation, advertised in recent years as the Segway® Human Transporter,‡ is designed as a monocycle seating one person. Lacking control, the motion of such a monocycle is unstable: it has to be stabilized by a system controlling the motion of the device.

In any practical system, control actions are limited in some way or another, so that an unstable object cannot be put into the necessary mode of operation from any state. In other words, the controllability domain, namely, the set of states from which, with the available control resources, the object may be brought to the desired mode of operation, occupies only part of the phase space. The domain of attraction of the desired mode of operation, resulting from the construction of a specific law of feedback control, is a subdomain of the controllability domain; most frequently, it occupies only part of the latter. In that situation it is an extremely important problem to maximize the domain of attraction for bounded control actions.

Examples of unstable controllable systems are a bicycle with a gyroscopic stabilizer [1] and a monocycle with a gryostabilizer [2] designed at the Moscow State University Institute of Mechanics.

Many unstable mechanical systems contain inverted pendulums as components. The control of a one-link pendulum and stabilization of its unstable upper equilibrium position are classical problems of theoretical mechanics and control theory. In most studies they are solved by displacing the point of suspension of the pendulum (see [3–5], etc.). The plane motion of a one-link pendulum with fixed suspension point has also been investigated theoretically and experimentally [6–8]. Attached to the end of such a pendulum is an electric motor with a flywheel. A control law has been constructed for the motor, under which the pendulum may be brought from any initial state to the unstable upper equilibrium position and stabilized there. The problem of flywheel stabilization has also been studied for an inverted pendulum on a cylinder [9]. Some problems touching upon the stability of motion of a one-wheel bicycle have also been considered [10].

The present paper is concerned with the forward motion of a cylinder (monocycle) with an inverted pendulum, rolling without slipping on a horizontal surface. The mechanical system studied has two degrees of freedom and one control input. The problem of controlling the unstable system presents the most difficulties when the number of control inputs is less than the number of degrees of freedom. A solution will be presented here of the problem of synthesizing a feedback control of bounded magnitude. We know of no publications in which such problems are solved for a monocycle.

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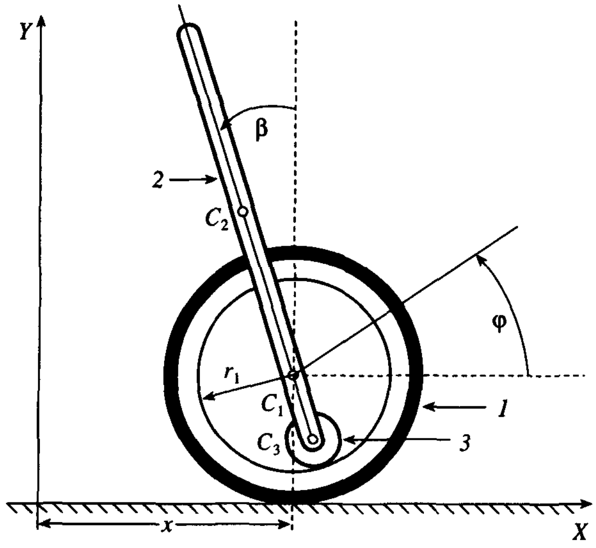

Fig. 1

1. THE MECHANICAL MODEL OF A MONOCYCLE

To investigate the forward motion of a monocycle, we consider a system consisting of three absolutely rigid bodies linked together by cylindrical hinges at axes C_1 and C_3 (Fig. 1). The axes of the hinges are perpendicular to the plane of the diagram. Body 1 is a cylinder, 2 is a pendulum, rigidly attached to which is the stator of an electric motor and 3 is a pinion, to which the rotor of the motor is rigidly attached. In the device called "Segway", body 2 also includes the person on the device. The centre of mass of body 2 is at the point C_2 and that of body 3 on its axis of rotation C_3. We let m_i denote the mass of body i ($i = 1, 2, 3$) and ρ_i its radius of inertia about the centre of mass C_i.

A cylinder of radius r, symmetrical about its axis C_1, can roll without slipping along a straight line on horizontal surface. Let φ denote the angle through which some fixed radius (marked on the cylinder), directed at the beginning of the motion along the horizontal axis X, rotates counter-clockwise; let x denote the displacement of the centre of mass of the cylinder, so that $\dot{x} = -\dot{\varphi}r$.

Pendulum 2, which may oscillate in a vertical plane, is linked by a hinge to axis C_1 of cylinder 1. Figure 1 provides a side view of this cylinder together with the inverted pendulum attached to it. Let β denote the angle of deflection of body 2, or more precisely, the angle of deflection of the straight line C_1C_2, from the vertical, measured counter-clockwise.

Rigidly attached to cylinder 1 is a hollow inner-tooth cylindrical surface of radius r_1 whose axis coincides with the axis C_1 of the cylinder. Pinion 3 of radius r_3 can roll around the inside of this toothed surface. When rolling, the pinion rotates freely around the C_3 axis, which is attached to the pendulum (Fig. 1). Attached to the pendulum is a DC electric motor whose stator, as already mentioned, is rigidly attached to the pendulum, and its rotor to pinion 3. To simplify matters, we will consider the case in which all the centres of mass C_i ($i = 1, 2, 3$) lie on the same straight line. Under these conditions, if $C_1C_2 = r_2$, $C_1C_3 = r_1 - r_3$, then $C_2C_3 = r_2 + r_1 - r_3$.

The mechanical system considered consists of three bodies and has two degrees of freedom, characterized by generalized coordinates φ and β or x and β. The variable φ is cyclic, and hence so is x. If this variable is ignored, the system has one and a half degrees of freedom. Setting up the equations of motion below, we shall make allowance for the variation of electric current in the rotor circuit. The charge is a cyclic coordinate; if it is ignored, the electromechanical system has two degrees of freedom.

With the generalized coordinates defined above, the expressions for the horizontal and vertical components V_{C_ix} and V_{C_iy} of the velocity vector V_{C_i} of the centre of mass C_i ($i = 1, 2, 3$) are

$$\begin{aligned}
&V_{C_1x} = \dot{x}, \quad V_{C_2x} = \dot{x} - \dot{\beta}r_2\cos\beta, \quad V_{C_3x} = \dot{x} + \dot{\beta}(r_1 - r_3)\cos\beta \\
&V_{C_1y} = 0, \quad V_{C_2y} = -\dot{\beta}r_2\sin\beta, \quad V_{C_3y} = \dot{\beta}(r_1 - r_3)\sin\beta
\end{aligned} \tag{1.1}$$

Since plane motion of the monocycle is being studied, the vectors of the absolute angular velocities of all three bodies are perpendicular to the vertical XY plane. The magnitudes Ω_i (i = 1, 2, 3) of these angular velocities are

$$\Omega_1 = \dot{\varphi} = -\frac{\dot{x}}{r}, \quad \Omega_2 = \dot{\beta}, \quad \Omega_3 = \left(\frac{r_3 - r_1}{r_3}\right)\dot{\beta} - \frac{r_1\dot{x}}{r_3 r} = -(\chi - 1)\dot{\beta} - \frac{\chi\dot{x}}{r} \tag{1.2}$$

where $\chi = r_1/r_3 > 1$ is the transmission ratio of the reductor.

Taking relations (1.1) and (1.2) into consideration, we obtain the following expression for the kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{3} m_i(V_{C_i}^2 + \rho_i^2\Omega_i^2) = \frac{1}{2}[a_{11}\dot{x}^2 + 2(a_{12} - a_{12c}\cos\beta)\dot{x}\dot{\beta} + a_{22}\dot{\beta}^2] \tag{1.3}$$

where

$$\begin{aligned} a_{11} &= m_1 + m_2 + m_3 + m_1\rho_1^2/r^2 + m_3\rho_3^2\chi^2/r^2, \quad a_{12} = m_3\rho_3^2\chi(\chi - 1)/r \\ a_{12c} &= m_2r_2 - m_3(r_1 - r_3), \quad a_{22} = m_2(r_2^2 + \rho_2^2) + m_3(\chi - 1)^2(r_3^2 + \rho_3^2) \end{aligned} \tag{1.4}$$

The potential energy is (g denotes the acceleration due to gravity)

$$\Pi = [m_2r_2 - m_3(r_1 - r_3)]g\cos\beta = a_{12c}g\cos\beta \tag{1.5}$$

No allowance will be made for the torque of the friction forces resisting the rolling of the cylinder. The influence of that torque on the motion of the system will be estimated below.

Considering the inverted pendulum, we shall assume that $r_2 > 0$ and, in addition, that

$$m_2r_2 > m_3(r_1 - r_3) \quad (a_{12c} > 0) \tag{1.6}$$

that is, the centre of mass of the pendulum together with body 3 is situated above the axis C_1 of the cylinder.

The magnetic energy of the system can be represented as

$$W = \frac{1}{2}\left[LI^2 + 2c\chi I\left(\frac{x}{r} + \beta\right)\right] \tag{1.7}$$

Where I is the current through the circuit of the rotor of the electric motor, L is the inductance of the circuit and c is the so-called coefficient of electromechanical interaction. Using expressions (1.3), (1.5) and (1.7), we can write the Lagrangian [11, 12] of the system as

$$\begin{aligned} \mathcal{L} = T - \Pi + W &= \frac{1}{2}[a_{11}\dot{x}^2 + 2(a_{12} - a_{12c}\cos\beta)\dot{x}\dot{\beta} + a_{22}\dot{\beta}^2] - \\ &- a_{12c}g\cos\beta + \frac{1}{2}\left[LI^2 + 2c\chi I\left(\frac{x}{r} + \beta\right)\right] \end{aligned} \tag{1.8}$$

2. THE EQUATIONS OF MOTION

Using Lagrange's method of the second kind [11, 12] and expression (1.8), we set up the equations of motion of the system

$$\begin{aligned} &a_{11}\ddot{x} + (a_{12} - a_{12c}\cos\beta)\ddot{\beta} + a_{12c}\sin\beta\dot{\beta}^2 = \frac{c\chi}{r}I \\ &(a_{12} - a_{12c}\cos\beta)\ddot{x} + a_{22}\ddot{\beta} - a_{12c}g\sin\beta = c\chi I \\ &L\dot{I} + RI + c\chi\left(\frac{\dot{x}}{r} + \dot{\beta}\right) = U \end{aligned} \tag{2.1}$$

where R is the ohmic resistance of the rotor circuit and U is the voltage applied to it, which plays the role of a generalized force. The quantity cI, proportion to the current I, describes the torque M of the electromagnetic forces acting between the stator and the rotor. The third equation of the system, Kirchhoff's equation, describes the transients in the rotor circuit.

The voltage U applied to the motor is bounded in absolute value:

$$|U(t)| \le U_0 \quad (U_0 = \text{const}) \tag{2.2}$$

Equations of the form (2.1) describe the motion of a cylinder with a pendulum suspended on its axis, whatever the design of the reductor of the control drive. The motion of the monocycle will be controlled by using internal forces. Under the influence of an internal moment, a relative displacement will be imparted to the pendulum and the cylinder. The external forces arising during this relative motion will be "organized" in such a way that the mechanism as a whole will move in the desired manner.

We introduce a dimensionless time variable τ by the formula $t = \vartheta\tau$, where $\vartheta = \sqrt{r/g}$; then Eqs (2.1) may be written in dimensionless variables as follows:

$$\begin{aligned} &\varphi'' + (j_1\cos\beta - j_2)\beta'' - j_1\sin\beta\beta'^2 = -i \\ &(j_1\cos\beta - j_2)\varphi'' + j_3\beta'' - j_1\sin\beta = i \\ &\theta i' + i + p(\beta' - \varphi') = u \end{aligned} \tag{2.3}$$

The prime denotes differentiation with respect to the dimensionless time τ. The five dimensionless parameters of the system are

$$j_1 = \frac{a_{12c}}{a_{11}r}, \quad j_2 = \frac{a_{12}}{a_{11}r}, \quad j_3 = \frac{a_{22}}{a_{11}r^2}, \quad \theta = \frac{L}{R\vartheta}, \quad p = \frac{c^2\chi^2\vartheta}{R_{a_{11}}r^2} \tag{2.4}$$

and the dimensionless current and voltage are

$$i = \frac{c\chi}{a_{11}rg}I, \quad u = \frac{c\ \ \chi}{Ra_{11}rg}U$$

Inequality (2.2) becomes, in the new notation,

$$|u| \le u_0, \quad u_0 = \frac{c\ \ \chi}{Ra_{11}rg}U_0 \tag{2.5}$$

If the inductance L in the rotor circuit (an electromagnetic constant of the time θ) is ignored, the order of the system of equations (2.1) is reduced by one. In dimensionless variables, one obtains instead of (2.3) the system

$$\begin{aligned} &\varphi'' + (j_1\cos\beta - j_2)\beta'' - j_1\sin\beta\beta'^2 = -u + p(\beta' - \varphi') \\ &(j_1\cos\beta - j_2)\varphi'' + j_3\beta'' - j_1\sin\beta = u - p(\beta' - \varphi') \end{aligned} \tag{2.6}$$

We introduce the notation

$$c_1 = c/R \text{ and } c_2 = c^2/R \tag{2.7}$$

Then

$$u = \frac{c\ \ \chi}{Ra_{11}rg}U = \frac{c_1\chi}{a_{11}rg}U, \quad u_0 = \frac{c_1\chi}{a_{11}rg}U_0, \quad p = \frac{c^2\chi^2\vartheta}{R_{a_{11}}r^2} = \frac{c_2\chi^2\vartheta}{a_{11}r^2} \tag{2.8}$$

If the inductance in the rotor winding is ignored, the torque M of the electromagnetic forces applied by the stator to the rotor is given by the expression [13]

$$M = c_1U - c_2\chi(\dot\beta - \dot\varphi) \tag{2.9}$$

The positive constant coefficient c_1 and c_2 (the back-emf coefficient) may be evaluated on the basis of the certified values of the starting and nominal torques, the nominal angular velocity and nominal voltage of the motor [13]. Knowing the values of c_1 and c_2, one can use formulae (2.7) to evaluate the coefficient c and the ohmic resistant R.

It shall be noted that the model (2.6) of the motion of a monocycle with inverted pendulum has much in common with the model of a flywheel-controlled pendulum [6–8]. Both models have a cyclic coordinate, they are of the same order, and each model has one stable and one unstable equilibrium position. As will be shown below, both models, linearized about the unstable equilibrium position, have one positive eigenvalue and two negative ones. These remarks account for the similarity of the methods and the results of investigating both systems.

3. STABILIZATION OF THE PENDULUM IN ITS UPPER UNSTABLE EQUILIBRIUM POSITION

In this section we shall investigate the simplified mathematical model (2.6), assuming that the time constant θ is negligibly small.

We will consider the problem of stabilizing the pendulum in its upper equilibrium position $\beta = 0$, on the assumption that it is already in the neighbourhood of the desired position at the beginning of the stabilization process. By this assumption, circular motions of the pendulum are excluded from consideration.

Linearized equations. We shall assume that, during the process of stabilizing the upper equilibrium position, the angle β and its derivative are close to zero. Then, linearizing Eqs (2.6), we obtain the system of equations

$$\varphi'' + j_4\beta'' = -u + p(\beta' - \varphi'), \quad j_4\varphi'' + j_3\beta'' - j_1\beta = u - p(\beta' - \varphi') \tag{3.1}$$

where $j_4 = j_1 - j_2$.

Since the angle of rotation φ of the cylinder is a cycle variable, we can introduce the angular velocity $\omega = \varphi'$ into the equations of motion (2.6) or (3.1), after which systems (2.6) and (3.1) become third-order systems. This substitution gives the linear equations (3.1) the form

$$\omega' + p\omega + j_4\beta'' - p\beta' = -u, \quad j_4\omega' - p\omega + j_3\beta'' + p\beta' - j_1\beta = u \tag{3.2}$$

When $u = 0$ the non-linear system (2.6) and the linear system (3.2) have the trivial solution

$$\omega = \varphi' = 0, \quad \beta = 0, \quad \beta' = 0 \tag{3.3}$$

corresponding to the vertical (unstable) equilibrium position of the pendulum and the cylinder at rest. The problem of stabilizing the equilibrium (3.3) will be considered later.

Solving Eqs (3.2) for the higher derivatives, we can write them in matrix notation as

$$z' = Bz + hu \tag{3.4}$$

$$z = \begin{Vmatrix} z_1 \\ z_2 \\ z_3 \end{Vmatrix} = \begin{Vmatrix} \omega \\ \beta \\ \beta' \end{Vmatrix}, \quad h = \begin{Vmatrix} h_1 \\ 0 \\ h_3 \end{Vmatrix}, \quad B = \|b_{lk}\| = \begin{Vmatrix} b_{11} & b_{12} & -b_{11} \\ 0 & 0 & 1 \\ b_{31} & b_{32} & -b_{31} \end{Vmatrix}, \quad l, k = 1, 2, 3$$

$$h_1 = -\frac{j_3 + j_4}{j_5}, \quad h_3 = \frac{j_4 + 1}{j_5}, \quad b_{11} = ph_1, \quad b_{12} = -\frac{j_1 j_4}{j_5}, \quad b_{31} = ph_3, \quad b_{32} = \frac{j_1}{j_5}, \quad j_5 = j_3 - j_4^2$$

The quantity j_5 is proportional to the determinant of the positive-definite matrix of the kinetic energy at $\beta = 0$, hence it is positive, which can also be verified directly by using expression (2.4) for the dimensionless parameters. The desired equilibrium position (3.3) in the new variables has the form $z = 0$. It follows from Kalman's criterion [14] that system (3.4) is completely controllable if and only if

$$\det\|h, Bh, B^2h\| = h_3^2(h_1 b_{32} - h_3 b_{12}) \neq 0 \tag{3.5}$$

Using the notations (1.4) and (2.4), it can be shown that $j_4 + 1 > 0$, and thus also $h_3 > 0$. The expression $h_1b_{32} - h_3b_{12}$, as is readily shown, vanishes if an only if $j_1(j_4 + 1) = 0$. Since $j_4 + 1 \neq 0$, inequality (3.5) fails to hold if and only if $j_1 = 0$, that is, when

$$a_{12c} = m_2r_2 - m_3(r_1 - r_3) = 0 \tag{3.6}$$

Equation (3.6) holds only in the case that the centre of mass of the pendulum together with body 3 lies on the axis C_1 of the cylinder. If condition (1.6) holds, system (3.4) is completely controllable.

Note that in case (3.6) both the linearized system (3.1) and the initial non-linear system (2.6) have the integral

$$\begin{gathered}[m_1(r^2 + \rho_1^2) + m_2r^2 + m_3(r^2 + \rho_3^2\chi)]\varphi' + \\ + [m_2(r_2^2 + \rho_2^2) + m_3r_3^2(1-\chi)^2 + m_3\rho_3^2(1-\chi)]\beta' = C = \text{const}\end{gathered}$$

which is an integral of the angular momentum of the system relative to the instantaneous centre (axis) of the velocities – the axis along which the cylinder is in contact with the supporting surface. Consequently, if condition (3.6) holds, not only the linear system (3.1) but also the non-linear system (2.6) are uncontrollable.

Eigenvalues of the open-loop system. We will find the position in the complex plane of the eigenvalues of the open-loop system obtained from (3.4) by putting $u = 0$ ($U = 0$), i.e., the eigenvalues of the matrix B. The characteristic equation of this third-order system has the form (μ is the spectral parameter)

$$F(\mu) = \mu^3j_5 + \mu^2p(1 + j_3 + 2j_4) - \mu j_1 - pj_1 = 0 \tag{3.7}$$

Taking $c_2 = 0$ as the origin (that is, assuming that there is no back-emf), we can then infer from relations (2.8) that $p = 0$. If condition (1.6) holds, then $j_1 > 0$ (see notation (2.4)). Equation (3.7) when $p = 0$ has two non-zero real roots, differing only in their signs, and one zero root:

$$\mu_1 = \sqrt{j_1/j_5}, \quad \mu_2 = 0, \quad \mu_3 = -\sqrt{j_1/j_5} \tag{3.8}$$

that is, the spectrum of the open-loop system at $p = 0$ is symmetrical about the imaginary axis. This is natural, since when $c_2 = 0$ the open-loop system is conservative. When the back-emf is "added" ($c_2 > 0$, $p > 0$), the zero eigenvalue is displaced to the left, and the two others are also displaced, but remain positive and negative for all values of $p > 0$. This statement holds because the function $F(\mu)$ changes sign three times as its argument μ varies from $-\infty$ to $+\infty$. Its graph intersects the negative μ axis twice and the positive axis once. Indeed,

$$F(-\infty) = -\infty < 0, \quad F(-p) = p^3(j_4 + 1)^2 > 0, \quad F(0) = -pj_1 < 0, \quad F(+\infty) = +\infty > 0$$

The inequality $F(-p) > 0$ is true, since $p > 0$ and $j_4 + 1 \neq 0$, but $F(0) < 0$, since $p > 0$ and $j_1 > 0$.

Thus, if condition (1.6) is satisfied, the characteristic equation (3.7) has three real roots – one positive ($\mu_1 > 0$) and two negative ($\mu_2, \mu_3 < 0$). Thus, the system under consideration, lacking control – that is, in the open-loop state – is unstable. The situation is analogous for a flywheel-controlled pendulum [6–8].

If the parameter p (the coefficient c_2) is small, approximate evaluation of the eigenvalues μ_l ($l = 1, 2, 3$) can make use of the expressions (3.8), which hold when $p = 0$ ($c_2 = 0$). In the linear approximation with respect to p (with respect to c_2), the expressions for μ_l ($l = 1, 2, 3$) take the form

$$\mu_1 = \sqrt{j_1/j_5} - p(1 + j_4)^2/(2j_5), \quad \mu_2 = -p, \quad \mu_3 = -\sqrt{j_1/j_5} - p(1 + j_4)^2/(2j_5)$$

Determination of the unstable coordinate and construction of the controllability domain. A linear transformation of the variables with constant non-singular matrix K

$$y = Kz \tag{3.9}$$

where $y = \|y_1y_2y_3\|^*$ (the asterisk denotes transposition), will bring system (3.4) to Jordan form: three scalar equations related to one another only by the control u

$$y_l' = \mu_l y_l + d_l u, \quad l = 1, 2, 3 \tag{3.10}$$

where

$$KB = \Lambda K, \quad \Lambda = \operatorname{diag}\|\mu_l\|, \quad d_l = [j_1 - \mu_l^2(j_3 + j_4)]/j_5$$

Since system (3.4) is controllable in Kalman's sense, it follows that $d_l \neq 0$ ($l = 1, 2, 3$). The elements of the matrix $K = \|k_{lk}\|$ ($l, k = 1, 2, 3$) can be evaluated using the relations

$$k_{l1} = \mu_l(b_{31} + \mu_l) - b_{32}, \quad k_{l2} = -b_{32}b_{11} + b_{12}(b_{31} + \mu_l), \quad k_{l3} = b_{12} - \mu_l b_{11} \tag{3.11}$$

The set of initial states in the space $Y(y_1, y_2, y_3)$ from which the system can be steered to the origin by a control voltage u for which the restriction (2.5) is satisfied is the strip [6–8, 15].

$$|y_1| < |d_1| u_0/\mu_1 \tag{3.12}$$

where

$$y_1 = k_{11}\omega + k_{12}\beta + k_{13}\beta' \tag{3.13}$$

and the elements k_{11}, k_{12}, k_{13} are evaluated by formulae (3.11) with $l = 1$. The set (3.12), which we denote by Q, is known as the controllability domain [15].

Using relations (3.12) and (3.13), we can write bounds for the initial values of each of the phase variables $\omega(0)$, $\beta(0)$, $\beta'(0)$, on the assumption that the two other variables vanish at the initial instant of time:

$$|\omega(0)| < \xi_1, \quad |\beta(0)| < \xi_2, \quad |\beta'(0)| < \xi_3; \quad \xi_l = \left|\frac{d_l}{k_{1l}}\right|\frac{u_0}{\mu_1}, \quad l = 1, 2, 3 \tag{3.14}$$

Synthesis of the stabilization law. When $u = 0$, the only "unstable" coordinate in system (3.10) is y_1. The instability can be suppressed using linear feedback of the form

$$u = \gamma y_1 \tag{3.15}$$

provided that the constant gain γ satisfies the inequality

$$\mu_1 + d_1\gamma < 0 \tag{3.16}$$

The positive eigenvalue μ_1 of the open-loop system (3.10) (when $u = 0$), when the system is closed by feedback (3.15), becomes a negative eigenvalue $\mu_1 = + d_1\gamma$. The negative eigenvalues μ_2 and μ_3 remain unchanged.

Taking into account the restriction on the control voltage, the feedback becomes

$$u = \begin{cases} -u_0 & \text{if} \quad \gamma y_1 \leq -u_0 \\ \gamma y_1 & \text{if} \quad |\gamma y_1| \leq u_0 \\ u_0 & \text{if} \quad \gamma y_1 \geq u_0 \end{cases} \tag{3.17}$$

The domain of attraction V of system (3.10), (3.17) is the whole controllability domain Q [6–8]. Thus the feedback (3.17) provides the maximum possible domain of attraction and is in that sense optimal.

The eigenvalues of the closed-loop system with feedback (3.15) are real, as are the eigenvalues of the open-loop system (3.10). Hence it follows that the transient in the closed-loop system with control (3.17) will be aperiodic for any initial conditions in the domain of attraction V. Thus, the method of stabilization described guarantees not only the largest possible zone of attraction (for the linearized system) but also a periodicity of the transient.

4. DECELERATION OF THE CYLINDER

Let us assume that the zero of the sensor of the angle β, through which the pendulum is deflected form the vertical, is displaced by a quantity $\Delta\beta$, and that this displacement remains constant throughout the control process ($\Delta\beta$ = const). When there is an error $\Delta\beta$, the control (3.17) becomes

$$u = \begin{cases} -u_0 & \text{if} \quad \gamma y_1^{\Delta} \le -u_0 \\ \gamma y_1^{\Delta} & \text{if} \quad |\gamma y_1^{\Delta}| \le u_0, \quad y_1^{\Delta} = k_{11}\omega + k_{12}(\beta + \Delta\beta) + k_{13}\beta' \\ u_0 & \text{if} \quad \gamma y_1^{\Delta} \ge u_0 \end{cases} \tag{4.1}$$

A steady-state solution of the non-linear system (2.6) and of the linear system (3.2), under the control (4.1), is described not by (3.3) but by the relations

$$\beta = \beta' = 0, \quad \omega = -\frac{u}{p} = -\frac{\gamma k_{12}\Delta\beta}{\gamma k_{11} + p} \quad (x' = -\omega r) \tag{4.2}$$

This solution is holds provided the control in the steady state is determined by the middle row of the formula (4.1). In other words, formulae (4.2) describe a steady-state solution if the displacement of zero $\Delta\beta$ is such that the value of the voltage u satisfies the restriction (2.5), that is

$$\left|\frac{p\gamma k_{12}\Delta\beta}{\gamma k_{11} + p}\right| \le u_0$$

It is important to note that, irrespective of the presence of the error in the sensor for the deflection β of the pendulum from the vertical, β will vanish in the steady state, just as when there is no such error. However, when that error occurs, the cylinder will not stop, but will continue to roll at constant angular velocity; the voltage in the steady state, like the angular velocity, will also be non-zero. As to the torque M of the electromagnetic forces operating between the stator and the rotor of the motor, it will vanish in the steady state (see (2.9)).

In order to stop the cylinder rolling in the steady state, one can augment the linear feedback signal (4.1) with a signal proportional, with some constant coefficient g, to the displacement $x(\tau)$ of the cylinder, a measured from a certain position x^*:

$$g[x(\tau) - x^*] = -gr\int_{\tau^*}^{\tau} \omega(\zeta)d\zeta \tag{4.3}$$

This can be verified theoretically using the non-linear system of equations (2.6) (or the linearized system (3.2)) with control (4.1). The signal (4.3) may be incorporated in the signal when a certain time τ^* has passed since the beginning of the control process, in which case one would set $x^* = x(\tau^*)$. In that case the set (3.12) will still be the domain of attraction. The signal (4.3) is proportional to the integral of the velocity of rotation of the cylinder. As is well known, when an integral term is incorporated into the feedback, the control process tends to become unstable. To avoid instability, the coefficient g should be chosen to be "not too" large. Admissible values of the coefficient from the standpoint of stability may be found using, for example, Hurwitz's criterion.

5. CONTROL OF THE MONOCYCLE MOTION

Let us assume now that the angle β is measured precisely and that the constant quantity $\Delta\beta$ is not a measurement error but a specially specified "setting" of the control law. Then, in the steady state, the cylinder will roll at a constant velocity, defined by the second expression in (4.2); the pendulum will be in the vertical position. Thus, having set the quantity $\Delta\beta$, one can control the motion of the cylinder. Under these conditions the feedback (4.1) obviously guarantees asymptotic stability of the motion (4.2).

Suppose now that the cylinder, while in motion, is subject to a resistance force proportional, say, to its velocity of motion, $f\dot{\varphi}$, where f = const > 0. This force may be due to the resistance of the air or to rolling friction. Then, in the first of equations (2.6), and also in the first of equations (3.2), one can add a term of the form $\sigma\omega$, where σ = const > 0 is a number proportional to the coefficient f. Given this force, the algebraic equations for the steady values of the coordinates ω, β are

$$\sigma\omega = -u - p\omega, \quad -j_1 \sin\beta = u + p\omega \tag{5.1}$$

It follows from these relations that

$$\sigma\omega = j_1 \sin\beta \tag{5.2}$$

Equation (5.2) shows that, when there is a force resisting rolling of the cylinder, the pendulum will be deflected from the vertical in the direction in which the cylinder is rolling, as is easily explained by physical considerations. Lacking such a resistance force, the pendulum will remain vertical (see formulae (4.42)).

Assuming that the angle β is small and using control (4.1) instead of (5.1), we obtain a system of algebraic equations

$$\sigma\omega = -\gamma[k_{11}\omega + k_{12}(\beta + \Delta\beta)] - p\omega, \quad \sigma\omega = j_1\beta$$

whose solution is

$$\omega = \frac{-j_1\gamma k_{12}\Delta\beta}{j_1(p + \gamma k_{11} + \sigma) + \gamma k_{12}\sigma}, \quad \beta = \frac{\sigma\omega}{j_1} \tag{5.3}$$

The voltage in the steady state is $u = -(\sigma + p)\omega$. If $\sigma = 0$ ($f = 0$), relations (5.3) reduce to (4.2).

6. NUMERICAL INVESTIGATIONS

Consider a device with the following parameter values

$$m_1 = 10 \text{ kg}, \quad r = 0.2 \text{ m}, \quad \rho_1 = 0.1 \text{ m}, \quad r_1 = 0.16 \text{ m}, \quad m_2 = 75 \text{ kg}, \quad r_2 = 0.9 \text{ m}, \quad \rho_2 = 0.3 \text{ m},$$

$$m_3 = 0.85 \text{ kg}, \quad r_3 = 0.04 \text{ m}, \quad \rho_3 = 0.03 \text{ m}, \quad \chi = 4,$$

$$c_1 = 0.6 \text{ N m/V}, \quad c_2 = 0.05 \text{ N m s}, \quad U_0 = 19 \text{ V}$$

With these parameter values, the positive eigenvalues is $\mu_1 = 0.834$ and the two other eigenvalues of the open-loop system are negative: $\mu_2 = -0.032$ and $\mu_3 = -0.995$. The maximum possible domain of attraction of the linearized system, with respect to the deflection angle of the pendulum (see the second inequality in (3.14)), is described by the inequality $|\beta(0)| < 18.2°$. The coefficients k_{11}, k_{12} and k_{13}, computed using formulae (3.11), take the values $k_{11} = -0.101$, $k_{12} = -2.58$, $k_{13} = -3.00$. Inequality (3.16) with these parameter values has the form $\gamma > 0.318$. Numerical investigations of non-linear system (2.6) have been carried out for an optimal control (3.17) with coefficients $\gamma k_{11} = -0.0695$, $\gamma k_{12} = -1.78$, $\gamma k_{13} = -2.06$, as obtained for $\gamma = 0.687$. The results show that the domain of attraction for the angle β is "almost" the same as for the linearized system: $|\beta(0)| < 17.8°$. The solid curves in Fig. 2 represent the transient in the variables ω, β (in degrees), x/r and u, obtained by integration of the non-linear equations (2.6) and (3.17) with initial data

$$\omega(0) = \beta'(0) = x(0) = 0, \quad \beta(0) = -10° \tag{6.1}$$

The solution of the system tends asymptotically to zero in the variables ω, β and u. The variable x cyclic; naturally, it does not tend to zero, in fact becoming quite "large" by the end of the transient. This is due to the fact that the duration of the transient is quite long, during which time the monocycle may have covered a substantial distance. The slow attention of the transient is explained by existence of the eigenvalue $\mu_2 = -0.032$, which is close to the imaginary axis. We recall that when the system is closed by feedback (3.15) (or (3.17)), the positive eigenvalue $\mu_1 = 0.834$ of the open-loop system is displaced to the left of the complex half-plane; the other two eigenvalues ($\mu_2 = -0.032$ and $\mu_3 = -0.995$) remain unchanged.

Thus, the control law (3.17), which maximizes the domain of attraction, turns out to be rather unsatisfactory from the standpoint of the attenuation time of the transient and the distance through which the monocycle will move while its steady state is being stabilized. If the control is augmented at some time after the beginning of the stabilization process by a term of the form (4.3), describing feedback depending on the variable x, this distance may be reduced, without at the same time reducing the domain of attraction. The duration of the transient and the distance itself can also can be reduced by relaxing the requirement that the feedback coefficients must have optimum values, but maintaining asymptotic stability of the steady state. Relaxation of that requirement, however, does reduce the domain of attraction.

Below we present the results of a numerical investigation of the non-linear system (2.6) with the following linear feedback with saturation

$$u = \begin{cases} -u_0 & \text{if} \quad \upsilon \le -u_0 \\ \upsilon & \text{if} \quad |\upsilon| \le u_0 \\ u_0 & \text{if} \quad \upsilon \ge u_0 \end{cases} \tag{6.2}$$

where $\nu = k_\omega \omega + k_\beta \beta + k_{\beta'}\beta'$ and the coefficients k_ω, k_β and $k_{\beta'}$ differ from the optimum coefficients. These coefficients are evaluated by designating the eigenvalues of the linear system (3.2) closed by the feedback $u = \upsilon$. If the close-loop system has, say, a three-fold eigenvalue, which we denote by μ_0, the expressions for these coefficients will be

$$k_\omega = -p + \frac{j_5\mu_0^3}{j_1}, \quad k_\beta = -\frac{3j_5\mu_0^2 + j_1}{j_4 + 1}, \quad k_{\beta'} = p - \frac{j_5}{j_4 + 1}\left(3 + \frac{j_3 + j_4}{j_1}\mu_0^2\right) \tag{6.3}$$

The dashed curves in Fig. 2 depict the transient in the variables ω, β, x/r and u with the same initial data (6.1) as before. This transient was obtained by integrating the non-linear system (2.6) with feedback (6.2). The feedback coefficients are evaluated by formulae (6.3) with $\mu_0 = -0.9$:

$$k_\omega = -0.916, \quad k_\beta = -3.13, \quad k_{\beta'} = -6.77$$

Examination of the dashed curves in Fig. 2 clearly shows that the transient is attenuated far more rapidly than in the case of optimum coefficients designated to maximize the domain of attraction; the distance through which the monocycle will move is reduced approximately to one fifth of its previous value. The domain of attraction as a function of the angle β is somewhat narrower than that obtained with the optimum values of coefficients: $|\beta(0)| < 16.2°$.

If the quantity $\Delta\beta$ in control law (4.1) is specified as a function of time, it follows from Section 5 that the displacement of the monocycle may be controlled. Let us consider the case in which $\Delta\beta$ is defined as a trapezoidal function of time:

$$\Delta\beta(\tau) = \begin{cases} k\tau, & 0 \le \tau \le \tau_1 \\ k\tau_1, & \tau_1 \le \tau \le \tau_2 \\ k(\tau_1 + \tau_2) - k\tau, & \tau_2 \le \tau \le \tau_1 + \tau_2 \\ 0, & \tau_1 + \tau_2 \le \tau \end{cases} \tag{6.4}$$

Figure 3 represents the solution of system (2.6), (4.1), (6.4) in the time interval $0 \le \tau \le 100$ with initial data

$$\omega(0) = \beta(0) = \beta'(0) = x(0) = 0$$

The parameters of the function (6.4) are chosen as follows: $k = 0.066$, $\tau_1 = 4$, $\tau_2 = 12$. This solution was constructed for optimum values of the feedback coefficients. Over the time interval $\tau_1 \le \tau \le \tau_2$ solution of the system tends to a steady motion in which $\beta = 0$ and the angular velocity $\omega = \text{const}$. If $\tau_1 + \tau_2 \le \tau$, the solution tends to the steady state $\beta = \omega = x' = 0$. By the time $\tau = 100$ ($t = 14.28$ sec) the monocycle has almost stopped, after moving a distance $x \approx 10$ m.

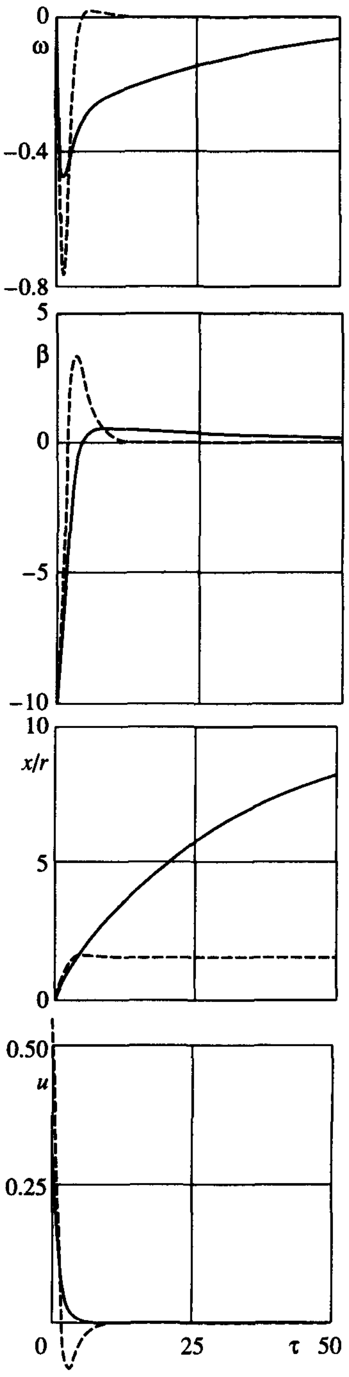

Fig. 2

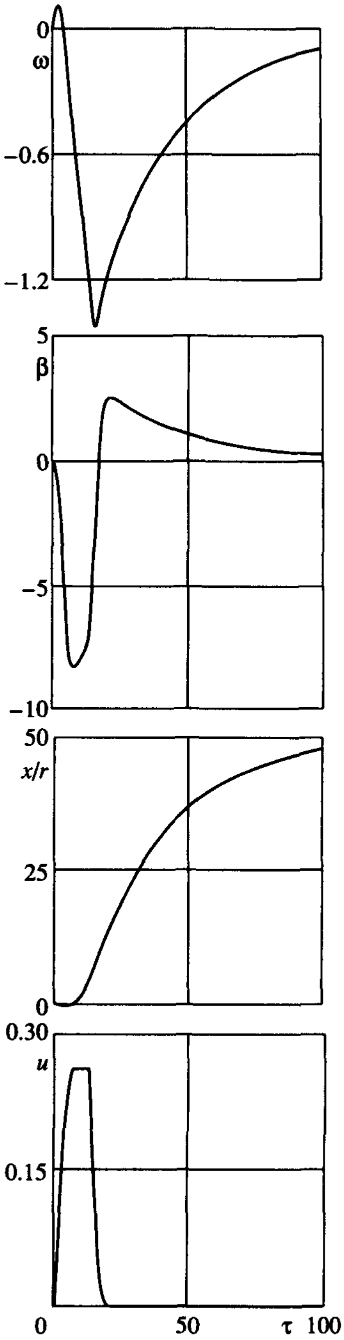

Fig. 3

7. EFFECT OF INDUCTANCE IN THE ROTOR CIRCUIT ON STABILITY

If there is an inductance L in the rotor circuit of the motor, transients will occur in the circuit, causing a delay in the control contour. In order to allow for such transients in the rotor circuit, equations (2.6) must be replaced by the complete equations (2.3), which include Kirchhoff's equation. System (2.3) when $u = 0$ and under control (3.17) has the solution

$$\omega = \varphi' = 0, \quad \beta = 0, \quad \beta' = 0, \quad i = 0 \tag{7.1}$$

Linearizing Eqs (2.3) about the solution (7.1) and replacing φ' by ω, we obtain the system

$$\begin{aligned} \omega' + j_4\beta'' &= -u + p(\beta' - \omega) \\ j_4\omega' + j_3\beta'' - j_1\beta &= u - p(\beta' - \omega) \\ \theta i' + i + p(\beta' - \omega) &= u \end{aligned} \tag{7.2}$$

which, under the control (3.17), also has the solution (7.1). The open-loop system(7.2) (i.e. with $u = 0$) has four eigenvalues, one of which is positive. At small values of θ, the other three are negative. Investigations show that as θ is increased two negative eigenvalues merge and then become complex conjugates.

If expression (3.15) for the linear feedback, with coefficients defined by (3.11), is substituted into (7.2), the results is a fourth-order system whose characteristic equation has the following form (μ is the spectral parameter)

$$\mu^4\theta j_5 + \mu^3 j_5 + \mu^2[p(1 + j_3 + 2j_4) - j_1\theta + j_5\gamma] + \mu j_1\left(\frac{p\gamma}{\mu_1^2} + \frac{\gamma}{\mu_1} - 1\right) + pj_1\left(\frac{\gamma}{\mu_1} - 1\right) = 0 \tag{7.3}$$

It follows from the condition that the coefficients of the polynomial (7.3) must be positive that the domain of asymptotic stability of solution (7.1) of system (7.3), (3.17) in the plane of the parameters θ, γ lies inside the angle

$$\theta < [j_5\gamma + (1 + j_3 + 2j_4)p]/j_1, \quad \gamma > \mu_1 \tag{7.4}$$

Moreover, it follows from the Hurwitz criterion that the domain of asymptotic stability lies beneath the curve

$$\theta = \frac{-a_1 + a_2\gamma + a_3\gamma^2}{\gamma(b_3\gamma - b_2)} \tag{7.5}$$

where

$$a_1 = (1 + j_4)^2\mu_1^4 p, \quad a_2 = \mu_1^2(p + \mu_1)[(1 + j_3 + 2j_4)p - j_5\mu_1]$$

$$a_3 = \mu_1^2(p + \mu_1)j_5, \quad b_2 = j_1\mu_1^2(p + \mu_1), \quad b_3 = j_1(p + \mu_1)^2$$

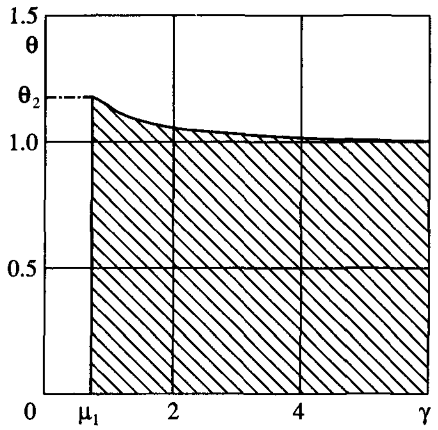

Fig. 4

The curve (7.5) has a horizontal asymptote

$$\theta = \theta_1 = \frac{\mu_1^2 j_5}{j_1(p+\mu_1)}$$

and consequently, if the delay is $\theta < \theta_1$ then, from the standpoint of asymptotic stability, any values of the gain γ are admissible. If $\theta > \theta_2$, where

$$\theta_2 = \theta_1 + \frac{\mu_1 p(1 + j_3 + 2j_4)}{j_1(p+\mu_1)}$$

the system is unstable, whatever the value of the gain γ.

Figure 4 shows the domain of asymptotic stability (the hatched area) in the θ, γ plane for the numerical parameter values specified above, as obtained from (7.4) and (7.5). With these parameter values, $\theta_1 = 0.97$, $\theta_2 = 1.20$.

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